Exercise 11D: Solutions

1 **a** $\sin(5\pi t)+\sin(\pi t)$

b $\frac{1}{2}(\sin 50^\circ-\sin 10^\circ)$

c $\sin(\pi x)+\sin\left(\frac{\pi x}{2}\right)$

d $\sin(A)+\sin(B+C)$

2 $\cos(\theta)-\cos(5\theta)$

3
$$\begin{aligned}&2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\\&=\sin\left(\frac{A-B+A+B}{2}\right)+\sin\left(\frac{A-B-A-B}{2}\right)\\&=\sin A+\sin(-B)\\&=\sin A-\sin B\end{aligned}$$

4
$$\begin{aligned}&2\sin(75^\circ)\sin(15^\circ)\\&=\cos(75-15)^\circ-\cos(75+15)^\circ\\&=\cos 60^\circ-\cos(90)^\circ\\&\therefore \sin(75^\circ)\sin(15^\circ)=\frac{1}{4}\end{aligned}$$

5 **a** $2\sin 39^\circ\cos 17^\circ$

b $2\cos 39^\circ\cos 17^\circ$

c $2\cos 39^\circ\sin 17^\circ$

d $-2\sin 39^\circ\sin 17^\circ$

6 **a** $2\sin(4A)\cos(2A)$

b $2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)$

c $2\sin\left(\frac{x}{2}\right)\cos\left(\frac{7x}{2}\right)$

d $-2\sin(2A)\sin(A)$

7
$$\begin{aligned}\text{LHS}&=\sin A+2\sin 3A+\sin 5A\\&=\sin A+\sin 3A+\sin 3A+\sin 5A\\&=2\sin 2A\cos(-A)+2\sin 4A\cos(-A)\\&=2\cos A(\sin 2A+\sin 4A)\\&=2\cos A(2\sin 3A\cos A)\\&=4\cos^2 A\sin 3A\\&=\text{RHS}\end{aligned}$$

8
$$\begin{aligned}\text{LHS}&=\sin(\alpha+\beta)\sin(\alpha-\beta)+\sin(\beta+\gamma)\sin(\beta-\gamma)+\sin(\gamma+\alpha)\sin(\gamma-\alpha)\\&=\frac{1}{2}(\cos(2\beta)-\cos(2\alpha)+\cos(2\gamma)-\cos(2\beta)+\cos(2\alpha)-\cos(2\gamma))\\&=0\\&=\text{RHS}\end{aligned}$$

9 $$\begin{aligned}\text{LHS} &= \cos 70^\circ + \sin 40^\circ \\ &= \cos 70^\circ + \cos 50^\circ \\ &= 2\cos(60^\circ \cos 10^\circ \\ &= \cos 10^\circ \\ &= \text{RHS}\end{aligned}$$

10 $$\begin{aligned}\text{LHS} &= \cos 20^\circ + \cos 100^\circ + \cos 140^\circ \\ &= \cos 20^\circ + 2\cos 120^\circ \cos(-20)^\circ \\ &= \cos 20^\circ - \cos(-20)^\circ \\ &= 0 \\ &= \text{RHS}\end{aligned}$$

11a $-\frac{5\pi}{6}, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$

b $-\pi, -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$

c $-\pi, -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$

d $-\pi, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

12a $$\begin{aligned}\cos 2\theta - \sin\theta &= 0 \\ 1 - 2\sin^2\theta - \sin\theta &= 0 \\ 2\sin^\theta + \sin\theta - 1 &= 0 \\ (2\sin\theta - 1)(\sin\theta + 1) &= 0 \\ \sin\theta = \frac{1}{2} \text{ or } \sin\theta &= -1 \\ \theta = \frac{\pi}{6} \text{ or } \theta &= \frac{5\pi}{6}\end{aligned}$$

b

$$\begin{aligned}\sin 5\theta - \sin 3\theta + \sin\theta &= 0 \\ 2\sin\theta\cos 4\theta + \sin\theta &= 0 \\ \sin\theta(\cos 4\theta + 1) &= 0 \\ \sin\theta = \text{ or } \cos 4\theta &= -1 \\ \theta = 0, \pi \text{ or } \theta &= \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}\end{aligned}$$

c $0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi$

d $\frac{\pi}{10}, \frac{\pi}{6}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{5\pi}{6}, \frac{9\pi}{10}$

13 $$\text{LHS} = \frac{\sin A + \sin B)}{\cos A + \cos B}$$
$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$
$$= \frac{2\sin\left(\frac{A+B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)}$$
$$= \tan\left(\frac{A+B}{2}\right)$$
$$= \text{RHS}$$

14 $$\begin{aligned}\text{LHS} &= 4\sin(A+B)\sin(B+C)\sin(C+A)\\ &= 2(\cos(A-C) - \cos(A+2B+C)\sin(C+A)\\ &= 2\cos(A-C)\sin(C+A) - 2\cos(A+2B+C)\sin(C+A)\\ &= \sin 2A + \sin 2C - (\sin(2A+2B+2C) + \sin(-2B))\\ &= \sin 2A + \sin 2C + \sin 2B - \sin(2A+2B+2C)\\ &= \text{RHS}\end{aligned}$$

15 $$\text{LHS} = \frac{\cos 2A - \cos 2B}{\sin(2A-2B)}$$
$$= \frac{2\sin\left(\frac{2A+2B}{2}\right)\sin\left(\frac{2B-2A}{2}\right)}{\sin(2A-2B)}$$
$$= \frac{2\sin(A+B)\sin(B-A)}{2\sin(A-B)\cos(A-B)}$$
$$= -\frac{\sin(A+B)}{\cos(A-B)}$$
$$= \text{RHS}$$

16

a $$\text{LHS} = \frac{\sin(A) + \sin(3A) + \sin(5A)}{\cos(A) + \cos(3A) + \cos(5A)}$$
$$= \frac{2\sin 3A\cos 2A + \sin 3A}{2\cos 3A\cos 2A + \cos 3A}$$
$$= \frac{\sin 3A(2\cos 2A + 1)}{\cos 3A(2\cos 2A + 1)}$$
$$= \tan 3A$$
$$= \text{RHS}$$

b $$\begin{aligned}\text{RHS} &= \cos(A+B)\cos(A-B)\\ &= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)\\ &= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B\\ &= \cos^2 A\cos^2 B - (1-\cos^2 A)(1-\cos^2 B)\\ &= \cos^2 A\cos^2 B - (1 - \cos^2 A - \cos^2 B + \cos^2 A\cos^2 B)\\ &= 1\cos^2 A + \cos^2 B - 1\\ &= \text{LHS}\end{aligned}$$

c $$\begin{aligned}\text{LHS} &= \cos^2(A-B) - \cos^2(A+B)\\ &= (\cos(A-B) - \cos(A+B))(\cos(A-B) + \cos(A+B))\\ &= 2\cos A\cos B \times 2\sin A\sin B\\ &= \sin 2A\sin 2B\\ &= \text{RHS}\end{aligned}$$

d $$\begin{aligned}\text{LHS} &= \cos^2(A-B) - \sin^2(A+B)\\ &= \cos^2(A-B) - (1 - \cos^2(A+B))\\ &= \cos 2A\cos 2B \text{ by } \{\backslash\text{bf 16b}\}\\ &= \text{RHS}\end{aligned}$$

17 $$\begin{aligned}\text{Let } S &= \sin x + \sin 3x + \sin 5x + \cdots + \sin 99x\\ \text{Then } 2\sin x S &= 2\sin^2 x + 2\sin x\sin 3x + 2\sin x\sin 5x + 2\sin x\sin 7x \cdots + 2\sin x\sin 99x\\ &= 2\sin^2 x + \cos 2x - \cos 4x + \cos 4x - \cos 6x + \cos 6x - \cos 8x \cdots + \cos 98x - \cos 100x\\ &= 2\sin^2 x + \cos 2x - \cos 100x\\ &= 2\sin^2 x + 1 - 2\sin^2 x - \cos 100x\\ &= 1 - \cos 100x\\ \therefore S &= \frac{1 - \cos 100x}{2\sin x}\end{aligned}$$